Several Constants Arising in Statistical Mechanics

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ABSTRACT. This is a brief survey of certain constants associated with random lattice models, including self-avoiding walks, polyominoes, the Lenz-Ising model, monomers and dimers, ice models, hard squares and hexagons, and percolation models.

1. Introduction

Random lattice models give rise to combinatorial problems that are often easily-stated but intractable. This article briefly presents several such problems, each involving the numerical estimation of asymptotic growth constants. Emphasis is given to recent developments. The author has been collecting and writing about various mathematical constants for five years. His large evolving website [18] provides further discussion and references; updates and corrections from readers are always welcome.

2. Self-Avoiding Walks

Let L denote the d-dimensional cubic lattice whose sites (vertices) are precisely all integer points in d-dimensional space. An n-step **self-avoiding walk** ω on L, beginning at the origin, is a sequence of sites $\omega(0)$, $\omega(1)$, $\omega(2)$, ..., $\omega(n)$ with $\omega(0) = 0$, $|\omega(j+1) - \omega(j)| = 1$ for all j and $\omega(i) \neq \omega(j)$ for all $i \neq j$.

The number of such walks ω is denoted by c(n). For example, c(0) = 1, c(1) = 2d and c(2) = 2d(2d - 1).

What can be said about the asymptotics of c(n)? It is known that

$$\mu = \lim_{n \to \infty} c(n)^{\frac{1}{n}}$$

exists and is nonzero. The value μ is called the **connective constant** and clearly depends on d. The current best rigorous lower and upper bounds for μ , plus the best known estimates for μ , are given in the following table [33], [1], [37], [15], [23]:

d	lower bound	best estimate	upper bound
2	2.62002	2.6381585	2.6939
3	4.572140	4.683907	4.7476
4	6.742945	6.7720	6.8179
5	8.828529	8.8386	8.8602
6	10.874038	10.8788	10.8886

The upper bounds were computed only recently [37] via the Goulden-Jackson cluster method [38]. Similar techniques can be used to estimate other constants associated with the combinatorics of words, e.g., the asymptotics of ternary square-free words and of binary cube-free words [18].

The connective constant values μ given above apply not only to the growth of the number of self-avoiding walks, but also to the growth of numbers of self-avoiding polygons and of self-avoiding walks with prescribed endpoints [27].

For d=2 and 3, there apparently is a positive constant γ such that

$$\lim_{n \to \infty} \frac{c(n)}{\mu^n \cdot n^{\gamma - 1}}$$

exists and is nonzero [33], [10], [12]. These constants are conjectured to be $\frac{43}{32}$ if d=2 and 1.1575... if d=3.

Another interesting object of study is the mean square displacement

$$s(n) = E\left\{\left|\omega(n)\right|^2\right\} = \frac{1}{c(n)} \cdot \sum_{\omega} \left|\omega(n)\right|^2$$

where the summation is over all *n*-step self-avoiding walks ω on L, weighted with uniform probability. Like c(n), it's believed for d=2 and 3 that there is a positive constant ν such that

$$\lim_{n \to \infty} \frac{s(n)}{n^{2\nu}}$$

exists and is nonzero; moreover [33], [10], [31] it's conjectured that $\nu = \frac{3}{4}$ if d = 2 and $\nu = 0.5877...$ if d = 3. The **critical exponents** γ and ν are thought to be *universal* in the sense that they are lattice-independent (although dimension-dependent). No one has yet discovered a proof of their existence on L, let alone a proof of universality.

3. Polyminoes

A domino is a pair of adjacent squares.

<FIGURE 1>

Generalizing, a **polyomino** or **lattice animal** of order n is a connected set of n adjacent squares, e.g., for n = 3,

Define A(n) to be the number of polyminoes of order n, where it's agreed that two polyminoes are distinct iff they have different shapes or different orientations. There

are different senses in which polyminoes are defined, e.g., free versus fixed, bond versus site, simply-connected versus not necessarily so, and others. For brevity's sake, we focus only on the fixed, site, possibly multiply-connected case.

Redelmeier [43] computed A(n) up to n = 24 and Conway and Guttmann [16] extended the sequence to n = 25. It is known that the limit

$$\alpha = \lim_{n \to \infty} A(n)^{\frac{1}{n}}$$

exists and is nonzero. The best known bounds on α are $3.791 \le \alpha \le 4.649551$ as discussed in [29], [42], [49], [21]. Improvements are possible using the new value A(25). The best known estimate, obtained via series expansion analysis by differential approximants [16], is $\alpha = 4.06265...$ A more precise asymptotic expression [16] for A(n) is

$$C \cdot n^{-1} \cdot \alpha^n$$

where C = 0.316..., but such an empirical result is far from being rigorously proved.

4. Lenz-Ising Model

Let L_N denote the regular d-dimensional cubic lattice with $N=n^d$ sites. For example, in two dimensions, L_N is the $n \times n$ square lattice with $N=n^2$. To eliminate boundary effects, L_N is wrapped around to form a d-dimensional torus so that, without exception, every site has 2d nearest neighbors.

Let's agree that a nonempty subgraph of L_N is connected and contains at least one bond (edge). Suppose that several subgraphs are drawn on L_N with the property that

- each bond of L_N is used at most once
- \bullet each site of L_N is used an *even* number of times (possibly zero).

Call such a configuration on L_N an even polygonal drawing.

<FIGURE 3>

Note that an even polygonal drawing is the union of simple, closed, bond-disjoint polygons but need not be connected. Other names in the literature for these configurations include closed or Eulerian subgraphs. Let B(r) be the number of even polygonal drawings for which there are exactly r bonds. For example, when d=2, it follows [13] that B(4)=N, B(6)=2N and $B(8)=\frac{1}{2}N(N+9)$. When d=3, it follows that B(4)=3N, B(6)=22N and $B(8)=\frac{1}{2}N(9N+375)$. Computing B(r)

for larger r is quite complicated, especially considering that in > 2 dimensions, the drawings can intertwine and be knotted!

Define the high temperature zero magnetic field free energy for the Ising model to be the series

$$\beta(z) = \lim_{n \to \infty} \frac{1}{N} \cdot \ln(1 + \sum_{r} B(r) \cdot z^{r}) = \sum_{k=0}^{\infty} \beta_{k} \cdot z^{k}$$

where z is called the **activity**. If T denotes temperature, then $\tanh^{-1}(z) \propto 1/T$. Coefficients β_k of this series, as polynomials in d, are given here [24]:

$$\begin{array}{ll} k & \beta_k \\ 4 & \frac{1}{2}d(d-1) \\ 6 & \frac{1}{3}d(d-1)(8d-13) \\ 8 & \frac{1}{4}d(d-1)(108d^2-424d+425) \\ 10 & \frac{2}{15}d(d-1)(2976d^3-19814d^2+44956d-34419) \end{array}$$

The radius of convergence [22], [9], [36], [20], [10] for $\beta(z)$ is

$$z_c = \lim_{k \to \infty} \beta_{2k}^{-\frac{1}{2k}} = \begin{cases} \sqrt{2} - 1 = 0.414213562373095049... & \text{if } d = 2\\ 0.218094... & \text{if } d = 3\\ 0.14855... & \text{if } d = 4\\ 0.1134... & \text{if } d = 5\\ 0.0920... & \text{if } d = 6\\ 0.0775... & \text{if } d = 7 \end{cases}$$

which is important since knowledge of z_c gives the *critical temperature* or *Curie point* T_c of the model. The exact two dimensional result is a famous outcome of work by Kramers and Wannier and by Onsager.

Here is a related problem. Suppose that several subgraphs are drawn on L_N with the property that

- each bond of L_N is used at most once
- all sites of L_N , except two, are even
- the two remaining sites are odd and must lie in the same (connected) subgraph.

Call this configuration an **odd polygonal drawing**. Note that an odd polygonal drawing is the bond-disjoint union of an even polygonal drawing and an (undirected) self-avoiding walk linking the two odd sites.

Enumerating odd polygonal drawings gives rise to what physicists call the **high tem**perature zero field magnetic susceptibility

$$\chi(z) = \sum_{k=0}^{\infty} \chi_k \cdot z^k$$

Coefficients χ_k of this series, as polynomials in d, are listed in [20]. The radius of convergence of $\chi(z)$ is the same as that for $\beta(z)$ for d > 1. Further, when d = 2,

$$\lim_{T \to T_c^+} \left(1 - \frac{T_c}{T} \right)^{\frac{7}{4}} \cdot \chi(z) = 0.9625817322...$$

Wu, McCoy, Tracy, Barouch [52], [19] determined an exact formula for this coefficient in terms of the Painlevé III function. The expression is complicated: can a simpler formula in terms of other mathematical constants (e.g., Glaisher's constant [18]) be found? An exact analog of this formula for d=3 is also evidently not known.

5. Monomers and Dimers

Let L_N be as before, but without wraparound. Two sites of L_N are called adjacent if the distance between them is 1. A **dimer** consists of two adjacent sites of L_N and the (non-oriented) bond connecting them. A **dimer arrangement** is a collection of disjoint dimers on L_N . Uncovered sites are called **monomers**, so dimer arrangements are also known as **monomer-dimer coverings**. A **dimer covering** is a dimer arrangement whose union contains all the sites of L_N .

For d = 2, let g(n) denote the number of distinct monomer-dimer coverings of L_N , then clearly g(1) = 1, g(2) = 7 and asymptotically [3], [25]

$$\kappa = \lim_{n \to \infty} g(n)^{\frac{1}{N}} = 1.940215351...$$

No exact expression for the constant κ is known. Baxter's approach for estimating κ was based on the corner transfer matrix variational approach. A natural way for physicists to discuss the monomer-dimer problem is to introduce an activity z for the number of monomers. The constant κ then corresponds to the situation in which z = 1. Values of q(n) were recently computed [26] up to n = 21.

The above contrasts with the special case of dimer coverings. An exact expression is known here for d = 2, due to Kastelyn, Fisher and Temperley. If f(n) is the number of distinct dimer coverings of L_N , then f(n) = 0 if n is odd and asymptotically [40]

$$\lim_{\substack{n \to \infty \\ n \text{ even}}} f(n)^{\frac{2}{N}} = \exp(\frac{2G}{\pi}) = 1.79162281206959342...$$

a fascinating and unexpected occurrence of Catalan's constant G. For d=3, the number h(n) of distinct dimer coverings [32] of L_N is h(2)=9, h(4)=5051532105 and asymptotically

$$\lim_{\substack{n \to \infty \\ n \text{ even}}} h(n)^{\frac{2}{N}} = \exp(\lambda)$$

where the constant λ is known only imprecisely. The current best rigorous bounds [41], [14], [44] are $0.44007584 \le \lambda \le 0.463107$ and the best known estimate [8] is $\lambda = 0.4466...$.

6. Ice Models

Let L_N be as before, with wraparound. An **orientation** of L_N is an assignment of a direction (arrow) to each bond of L_N . Assume that d=2 henceforth. What is the number, $\theta(n)$, of orientations of L_N such that at each site there are exactly two inward and two outward pointing edges? (Such orientations are said to obey the **ice** rule and are also called **Eulerian orientations**.) Here is a sample configuration:

After intricate analysis, Lieb proved that [40], [45]

$$\lim_{n \to \infty} \theta(n)^{\frac{1}{N}} = \left(\frac{4}{3}\right)^{\frac{3}{2}} = 1.539600717839002039...$$

This constant is known as the **residual entropy for square ice**. Interestingly, $3\theta(n)$ is also the number of ways of coloring the square faces of L_N with three colors so that no two adjacent faces are colored alike [4]. Values of $\theta(n)$ were recently computed [26] up to n = 13, and relevant computational complexity issues were discussed in [35].

The residual entropy W for ordinary hexagonal ice Ice-Ih and for cubic ice Ice-Ic (both complicated three-dimensional lattices) satisfy [39]

and are equal within the limits of Nagle's estimation error. These configurations are not the same as the simple models mathematicians tend to focus on. It would be interesting to see the value of W for the customary $n \times n \times n$ cubic lattice L_3 , either with the ice rule in effect (two arrows point out and four arrows point in) or with Eulerian orientation (three arrows point out and three arrows point in). No one appears to have done this.

7. HARD SQUARES AND HEXAGONS

Consider the set of all $n \times n$ binary matrices. What is the number F(n) of such matrices with no pairs of adjacent 1's? Two 1's are said to be adjacent if they lie in positions (i, j) and (i + 1, j), or if they lie in positions (i, j) and (i, j + 1), for some i, j. Equivalently, F(n) is the number of configurations of non-attacking Princes on an $n \times n$ chessboard, where a "Prince" attacks the four adjacent, non-diagonal places. Let $N = n^2$, then

$$\xi = \lim_{n \to \infty} F(n)^{\frac{1}{N}} = 1.50304808247533226...$$

is the hard square entropy constant [6], [11], [34], [5]. Essentially nothing is known about the arithmetic character of ξ .

Instead of an $n \times n$ binary matrix, consider an $n \times n$ binary array which looks like:

$$\begin{pmatrix} a_{11} & a_{23} \\ & a_{22} & a_{34} \\ a_{21} & a_{33} \\ & a_{32} & a_{44} \\ a_{31} & a_{43} \\ & a_{42} & a_{54} \\ a_{41} & a_{53} \\ & a_{52} & a_{64} \end{pmatrix}$$

(here n = 4). What is the number G(n) of such arrays with no pairs of adjacent 1's? Two 1's here are said to be adjacent if they lie in positions (i, j) and (i + 1, j), or in (i, j) and (i, j + 1), or in (i, j) and (i + 1, j + 1), for some i, j. Equivalently, G(n) is the number of configurations of non-attacking Kings on an $n \times n$ chessboard with regular hexagonal cells. It's surprising that the **hard hexagon entropy constant**

$$\eta = \lim_{n \to \infty} G(n)^{\frac{1}{N}} = 1.395485972479302735...$$

is algebraic (in fact, is solvable in radicals [28]) with minimal integer polynomial [55]

 $25937424601x^{24} + 2013290651222784x^{22} + 2505062311720673792x^{20} + \\797726698866658379776x^{18} + 7449488310131083100160x^{16} + \\2958015038376958230528x^{14} - 72405670285649161617408x^{12} + \\107155448150443388043264x^{10} - 71220809441400405884928x^{8} \\ -73347491183630103871488x^{6} + 97143135277377575190528x^{4} \\ -32751691810479015985152$

This is a consequence of Baxter's exact solution of the hard hexagon model [4], [2] via theta elliptic functions and the Rogers-Ramanujan identities from number theory!

Just as series for the Ising model were defined using counts of even polygonal drawings with exactly r bonds, series for the hard hexagon model can be defined using counts of non-attacking configurations of exactly r Kings (and likewise for the hard square model). The radius of convergence for the hexagon series

$$z_c = \frac{11 + 5\sqrt{5}}{2} = 11.09016994374947424...$$

possesses an exact expression. No similar theoretical breakthrough has occurred for the square model, hence the radius of convergence for the square series

$$z_c = 3.7962...$$

has no known analogous formula [4]. These values are important since they correspond to the critical activity (e.g., temperature or density) at which a phase transition occurs in the model.

If one replaces Princes by Kings on the chessboard with square cells, then the corresponding constant [34] is 1.342643951124... A related problem, that of enumerating the *maximal* configurations of $\frac{N}{4}$ nonattacking Kings, was discussed in [51], [30].

8. Percolation Models

Let M be a random $n \times n$ binary matrix satisfying

- $m_{ij} = 1$ with probability p, 0 with probability 1 p for each i, j
- m_{ij} and m_{kl} are independent for all $(i, j) \neq (k, l)$.

An **s-cluster** is an isolated grouping of s adjacent 1's in M, where adjacency means horizontal or vertical neighbors (not diagonal). For example, the 4×4 matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

has one 1-cluster, two 2-clusters and one 4-cluster. The total number of clusters is 4 in this case. For arbitrary n, the total cluster count is a random variable with normalized expected value

$$K_n = \frac{E(C_n)}{n^2}$$

The limit $K_S(p)$ of K_n exists as n approaches infinity, and $K_S(p)$ is called the **mean** cluster count per site or the **mean** cluster density for the site percolation

model. It's known that $K_S(p)$ is twice continuously differentiable on [0,1]; further, $K_S(p)$ is analytic on [0,1] except possibly at one point $p = p_c$. Monte Carlo simulation and numerical Padé approximants can be used to compute $K_S(p)$. For example [54], it's known that $K_S(\frac{1}{2}) = 0.065770...$

Instead of an $n \times n$ binary matrix M, consider a binary array A of 2n(n-1) entries which looks like

$$A = \begin{pmatrix} a_{12} & a_{14} & a_{16} \\ a_{11} & a_{13} & a_{15} & a_{17} \\ a_{22} & a_{24} & a_{26} \\ a_{21} & a_{23} & a_{25} & a_{27} \\ a_{32} & a_{34} & a_{36} \\ a_{31} & a_{33} & a_{35} & a_{37} \\ a_{42} & a_{44} & a_{46} \end{pmatrix}$$

(here n = 4). One should associate a_{ij} not with a site of the $n \times n$ square lattice (as one does for m_{ij}) but with a bond. An s-cluster here is an isolated, connected subgraph of the graph of all bonds associated with 1's. For example, the array

$$A = \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

has one 1-cluster, one 2-cluster and one 4-cluster. For **bond percolation models** such as this, one often includes 0-clusters in the total count as well, that is, isolated sites with no attached 1's bonds. In this case there are seven 0-clusters, hence the total number of clusters C_4 is 10. The limiting mean cluster density

$$K_B(p) = \lim_{n \to \infty} K_n = \lim_{n \to \infty} \frac{E(C_n)}{n^2}$$

exists as n approaches infinity and similar smoothness properties hold. Remarkably, however, an exact integral expression exists at $p = \frac{1}{2}$ for the limiting mean cluster density [47], [17]

$$K_B(\frac{1}{2}) = -\frac{1}{8}\cot(y) \cdot \frac{d}{dy} \left\{ \frac{1}{y} \cdot \int_{-\infty}^{\infty} \operatorname{sech}\left(\frac{\pi x}{2y}\right) \cdot \ln\left(\frac{\cosh(x) - \cos(2y)}{\cosh(x) - 1}\right) dx \right\} \bigg|_{y = \frac{\pi}{2}}$$

which Adamchik [54] recently simplified to

$$K_B(\frac{1}{2}) = \frac{3\sqrt{3} - 5}{2} = 0.09807621135331594...$$

An analogous expression for the limiting variance of cluster density was computed in [54]. In the same way, the bond percolation model on the triangular lattice gives a known limiting mean cluster density at a specific value of p (discussed below), but the relevant variance is not known here.

Let's turn attention away from mean cluster density K(p) and instead toward mean cluster size S(p). In the examples given earlier, $S_4 = (1+2+2+4)/4 = 9/4$ for the site case, $S_4 = (1+2+4)/3 = 7/3$ for the bond case, and S(p) is the limiting value of $E(S_n)$ as n approaches infinity. The **critical probability** or **percolation threshold** p_c is defined to be [46], [27]

$$p_c = \inf_{\substack{0$$

that is, the concentration at which an infinite cluster appears in the infinite lattice. For site percolation on the square lattice, there are rigorous bounds [50], [48]

$$0.556 < p_c < 0.679492$$

and an estimate [53] $p_c = 0.5927460...$ based on extensive simulation. Ziff [54] has additionally calculated that $K_S(p_c) = 0.0275981...$ via simulation.

In contrast, for bond percolation on the square and triangular lattices, there are exact results due to Sykes and Essam. Keston [27] proved that $p_c = \frac{1}{2}$ on the square lattice, corresponding to the expression $K_B(\frac{1}{2})$ above. On the triangular lattice, Wierman [27] proved that

$$p_c = 2 \cdot \sin\left(\frac{\pi}{18}\right) = 0.347296355333860698...$$

corresponding to another exact expression [7], [54]

$$K_B(p_c) = \frac{35}{4} - \frac{3}{2} \cdot \csc\left(\frac{\pi}{18}\right) = \frac{23}{4} - \frac{3}{2} \cdot \left\{\sqrt[3]{4 \cdot \left(1 + i\sqrt{3}\right)} + \sqrt[3]{4 \cdot \left(1 - i\sqrt{3}\right)}\right\}$$

$$= 0.1118442752845497...$$

Similar results apply for the hexagonal lattice by duality.

Much energy has been placed into the computation of universal exponents for random lattice models in d-dimensional space (akin to what we briefly mentioned for self-avoiding walks). The existence of such exponents is hypothesized on the basis of both theoretical procedures (finite size scaling and renormalization group methods) and experimental data. We shall not attempt to discuss this important subject, but instead refer readers to [33], [46], [27].

References

- [1] Alm, S. E., Upper bounds for the connective constant of self-avoiding walks, *Combin. Probab. Comput.* 2 (1993) 115-136.
- [2] Andrews, G. E., The reasonable and unreasonable effectiveness of number theory in statistical mechanics, *Proc. Symp. Applied Math.* 46, ed. S. A. Burr (Orono conf., 1991) Amer. Math. Soc., 1992, 21-34.
- [3] Baxter, R. J., Dimers on a rectangular lattice, J. Math. Physics 9 (1968) 650-654.
- [4] Baxter, R. J., Exactly Solved Models in Statistical Mechanics, Academic Press 1982.
- [5] Baxter, R. J., Planar lattice gases with nearest-neighbour exclusion, *Annals Combin.* 3 (1999).
- [6] Baxter, R. J., Enting, I. G., Tsang, S. K., Hard-square lattice gas, J. Stat. Phys. 22 (1980) 465-489.
- [7] Baxter, R. J., Temperley, H. N. V., Ashley, S. E., Triangular Potts model at its transition temperature, and related models, *Proc. Royal Soc. London A* 358 (1978) 535-559.
- [8] Beichl, I., Sullivan, F., Approximating the permanent via importance sampling with application to the dimer covering problem, *J. Comput. Phys.*, submitted.
- [9] Blöte, H. W. J., Luijten, E., Heringa, J. R., Ising universality in three dimensions: a Monte Carlo study, *J. Phys. Math. A* 28 (1995) 6289-6313.
- [10] Butera, P., Comi, M., N-vector spin models on the simple-cubic and the body-centered-cubic lattices: a study of the critical behavior of the susceptibility and of the correlation length by high-temperature series extended to order 21, *Phys. Rev. B* 56 (1997) 8212-8240.
- [11] Calkin, N. J., Wilf, H. S., The number of independent sets in a grid graph, SIAM J. Discrete Math. 11 (1998) 54 60.
- [12] Caracciolo, S., Causo, M. S., Pelissetto, A., Monte Carlo results for three-dimensional self-avoiding walks, *Nucl. Phys. Proc. Suppl.* 63 (1998) 652-654.
- [13] Cipra, B. A., An introduction to the Ising model, Amer. Math. Monthly 94 (1987) 937-959.

- [14] Ciucu, M., An improved upper bound for the three dimensional dimer problem, *Duke Math. J.*, to appear.
- [15] Conway, A. R., Guttman, A. J., Lower bound on the connective constant for square lattice self-avoiding walks, *J. Phys. A*. 26 (1993) 3719-3724.
- [16] Conway, A. R., Guttmann, A. J., On two-dimensional percolation, J. Phys. A 28 (1995) 891-904.
- [17] Essam, J. W., Percolation and cluster size, *Phase Transitions and Critical Phenomena*, vol. II, ed. C. Domb and M. S. Green, Academic Press 1972, 197-270.
- [18] Finch, S. R., Favorite Mathematical Constants, MathSoft Inc., website URL http://www.mathsoft.com/asolve/constant/constant.html, 1998.
- [19] Gartenhaus, S, McCullough, W. S., Higher order corrections for the quadratic Ising lattice susceptibility at criticality, *Phys. Rev. B* 38 (1988) 11688-11703.
- [20] Gofman, M., Adler, J., Aharony, A., Harris, A. B., Stauffer, D., Series and Monte Carlo study of high-dimensional Ising models, J. Stat. Phys. 71 (1993) 1221-1230.
- [21] Guttmann, A. G., On the number of lattice animals embeddable in the square lattice, J. Phys. A 15 (1982) 1987-1990.
- [22] Guttmann, A. J., Enting, I. G., The high-temperature specific heat exponent of the 3D Ising model, *J. Phys. A* 27 (1994) 8007-8010.
- [23] Hara, T., Slade, G., Sokal, A. D., New lower bounds on the self-avoiding-walk connective constant, J. Stat. Phys. 72 (1993) 479-517; erratum, 78 (1995) 1187-1188.
- [24] Harris, A. B., Meir, Y., Recursive enumeration of clusters in general dimension on hypercubic lattices, *Phys. Rev. A* 36 (1987) 1840-1848.
- [25] Heise, M., Upper and lower bounds for the partition function of lattice models, *Physica A* 157 (1989) 983-999.
- [26] Henry, J. J., private communications (1997-1998).
- [27] Hughes, B. D., Random Walks and Random Environments, vols. 1 and 2, Oxford, 1996.
- [28] Joyce, G. S., On the hard hexagon model and the theory of modular functions, *Phil. Trans. Royal Soc. London A* 325 (1988) 643-702.

- [29] Klarner, D. A., Rivest, R. L., A procedure for improving the upper bound for the number of n-ominoes, *Canad. J. Math.* 25 (1973) 585-602.
- [30] Larsen, M., The problem of kings, Elec. J. Comb. 2 (1995).
- [31] Li, B., Madras, N., Sokal, A. D., Critical exponents, hyperscaling and universal amplitude ratios for two- and three-dimensional self-avoiding walks, J. Stat. Phys. 80 (1995) 661-754.
- [32] Lundow, Per Håkan, Computation of matching polynomials and the number of 1-factors in polygraphs, Umeå University Math. Dept. preprint 12-1996 (1996).
- [33] Madras, N., Slade, G., The Self-Avoiding Walk, Birkhäuser, 1993.
- [34] McKay, B. D., private communication (1996).
- [35] Mihail, M., Winkler, P., On the number of Eulerian orientations of a graph, Proc. Third Annual ACM-SIAM Symposium on Discrete Algorithms, Orlando FL., 1992, pp. 138-145; also appears in Algorithmica 16 (1996) 402-414.
- [36] Münkel, C., Heermann, D. W., Adler, J., Gofman, M., Stauffer, D., The dynamical critical exponent of the two-, three- and five-dimensional kinetic Ising model, *Physica A* 193 (1993) 540-552.
- [37] Noonan, J., New upper bounds for the connective constants of self-avoiding walks, J. Stat. Phys. 91 (1998) 871-888.
- [38] Noonan, J., Zeilberger, D., The Goulden-Jackson cluster method: extensions, applications and implementations, J. Difference Eq. Appl., to appear.
- [39] Nagle, J. F., Lattice statistics of hydrogen bonded crystals: I. The residual entropy of ice, J. Math. Phys. 7 (1966) 1484-1491.
- [40] Percus, J. K., Combinatorial Methods, Springer-Verlag 1971.
- [41] Priezzhev, V. B., The statistics of dimers on a three-dimensional lattice, II. An improved lower bound, J. Stat. Phy. 26 (1981) 829-837.
- [42] Rands, B. M. I., Welsh, D. J. A., Animals, trees and renewal sequences, *IMA J. Appl. Math.* 27 (1981) 1-17.
- [43] Redelmeier, D. H., Counting polyominoes: Yet another attack, *Discrete Math.* 36 (1981) 191-203.

- [44] Schrijver, A., Counting 1-factors in regular bipartite graphs, J. Combin. Theory B 72 (1998) 122-135; also MR 82a:15004.
- [45] Stanley, R. P., Enumerative Combinatorics, vol. 1, Cambridge Univ. Press, 1997.
- [46] Stauffer, D., Aharony, A., Introduction to Percolation Theory, 2nd ed., Taylor and Francis, 1992.
- [47] Temperley, H. N. V., Lieb, E. H., Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices; some exact results for the 'percolation' problem, *Proc. Royal Soc. London A* 322 (1971) 251-280.
- [48] van den Berg, J., Ermakov, A., A new lower bound for the critical probability of site percolation on the square lattice, *Random Structures and Algorithms* 8 (1996) 199-212.
- [49] Whittington, S. G., Soteros, C. E., Lattice animals: Rigorous results and wild guesses, in *Disorder in Physical Systems: A Volume in Honour of J. M. Hammersley*, ed. G. R. Grimmett and D. J. A. Welsh, Oxford, 1990.
- [50] Wierman, J. C., Substitution method critical probability bounds for the square lattice site percolation model, *Combin. Probab. Comput.* 4 (1995) 181-188.
- [51] Wilf, H. S., The problem of kings, *Elec. J. Comb.* 2 (1995).
- [52] Wu, T. T., McCoy, B. M., Tracy, C. A., Barouch, E., Spin-spin correlation functions for the two-dimensional Ising model: exact theory in the scaling region, *Phys. Rev. B* 13 (1976) 316-374.
- [53] Ziff, R. M., Spanning probability in 2D percolation, Phys. Rev. Letters 69 (1992) 2670-2673.
- [54] Ziff, R.M., Finch, S. R., Adamchik, V., Universality of finite-size corrections to the number of critical percolation clusters, *Phys. Rev. Lett.* 79 (1997) 3447-3450.
- [55] Zimmermann, P., private communication (1996).

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